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Super connected and hyper connected arithmetic graphs

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Abstract
A connected graph G is said to be super connected if every minimum vertex-cut isolates a vertex of G. Moreover a graph is said to be hyper connected if for every minimum vertex cut S, $G-S$ has exactly two components, one of which is an isolated vertex. In this paper, we focussed the concept in arithmetic graphs V_n and proved that, for an arithmetic graph $G=V_n$, $n=P_1^{a_1}\times P_2^{a_2}\times .... \times P_r^{a_r}$ where $0<a_i\leq 2$ and $r>3$ is super and hyper connected and for at least one $a_i\geq 3$, the graph $G=V_n$ is only super connected. Also, it is clear that for every arithmetic graph $G=V_n$, n is any integer is super and hyper edge connected.

Keywords
Arithmetic graph, super connected, hyper connected, super connectivity, hyper connectivity.

AMS Subject Classification 05C40.

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Contents

1. Introduction

For notation and graph theory terminology not given here, we follow [2]. In [4] the connectivity number of an arithmetic graph is studied by L. Mary Jenitha and S. Sujitha. Later the average connectivity of the same graph is discussed by the same authors in[5]. The super connected and hyper connected definitions are from [8] and authors used the concept for studying line graphs. In this paper we studied the concepts super connected, hyper connected, super connectivity, hyper connectivity of an arithmetic graph. All graphs in this paper considered are simple graphs. A connected graph G is $super\ connected (super\ edge\ connected)$ if every minimum vertex -cut (edge-cut) S of G,$G-S$ has isolated vertices. The cardinality of the minimum vertex cut is the $super\ connected\ number$ $(super\ edge\ connected\ number)$ and is denoted by $\kappa_s(G)(\lambda_s(G))$. A connected graph G is $hyper\ connected$ $(hyper\ edge\ connected)$ if every minimum vertex -cut(edge-cut) S of G,$G-S$ has exactly two components, one of which is an isolated vertex. The cardinality of the minimum vertex cut is the $hyper\ connected\ number$ $(hyper\ edge\ connected\ number)$ and is denoted by $\kappa_h(G)$ $(\lambda_h(G))$. A subset $S\subset V(G)$ is called a $super-vertex\ cut$ $(super-edge\ cut)$ if $G-S$ is not connected and every component contains at least two vertices . The $super\ connectivity$ $\kappa_{sc}(G)$ is the minimum cardinality over all super vertex cuts in G.In general, super vertex -cuts (respectively super edge-cuts)in G do not always exist then the number is denoted by ∞.This concept is studied from [3]. Through out the article we used the notation for an arithmetic graph as $G=V_n, n=P_1^{a_1}\times P_2^{a_2}\times .... \times P_r^{a_r}$. Some authors used the notation as $G=V_n$, $n=P_1^{a_1}P_2^{a_2}....P_r^{a_r}$ for an arithmetic graph. The following theorems are used in sequel.

Theorem 1.1. [2] *For a connected graph* $G, \kappa\leq\kappa^{'}\leq\delta$.

Theorem 1.2. [2] *A vertex v of a tree G is a cut vertex of G if and only if $d(v)>1$.*

Theorem 1.3. [6] *For an arithmetic graph* $G=V_n$, $n=P_1^{a_1}\times P_2^{a_2}\times .... \times P_r^{a_r}$,

$$\delta(G)=\begin{cases} r, & r\geq 3\\ 1, & r=2\end{cases}$$

Theorem 1.4. [4]*For an arithmetic graph* $G=V_n$, $n=p_1^{a_1}\times p_2^{a_2}$ *where p_1 and p_2 are distinct primes, then* $\kappa(V_n)=$

$$\kappa^{'}(V_n)=\begin{cases} 1\ for\ a_i=1\ and\ a_j>1; i,j=1,2\\ 2\ for\ a_i>1; i=1,2\end{cases}$$

Theorem 1.5. [5]*For an arithmetic graph* G=V_n, $n=p_1^{a_1}\times$

$p_2^{a_2}$ *where* p_1 *and* p_2 *are distinct primes,*$a_1, a_2 \geq 1$*then* $\varepsilon = 4a_1a_2 - a_1 - a_2$, *where* ε *is the size of the graph* G.

2. Super connected and Hyper connected Arithmetic Graphs

In this section we identified super connected and hyper connected arithmetic graphs from the class of arithmetic graphs and their corresponding numbers.

Theorem 2.1. *For an arithmetic graph* $G = V_n, n = p_1 \times p_2 \times \ldots \times p_r$ *is super and hyper connected and*

$$\kappa_s(G) = \kappa_h(G) = \begin{cases} 1, \text{ if } r = 2 \\ r, \text{ if } r > 2 \end{cases}$$

Proof. **Case (i)** when $r = 2$
In this case G is a tree with an internal vertex and two end vertices. Let p_1, p_2 be the end vertices and $p_1 \times p_2$ be an internal vertex of G. By theorem 1.2, no end vertex of G is a cut vertex of G and every internal vertices of G are cut vertices. Therefore the vertex cut S(say) contains the internal vertex $p_1 \times p_2$ only. Clearly the graph $G - S$ contains exactly two components namely the isolated vertices p_1 and p_2. Therefore by definition, the graph G is hyper connected and super connected. Hence the super connected number $\kappa_s(G) = \kappa_h(G) = |S| = 1$.

Case (ii) when $r = 3$
In this case the arithmetic graph is $G = V_n$ where $n = p_1 \times p_2 \times p_3$. Let the vertices of G be $\{p_1, p_2, p_3, p_1 \times p_2, p_1 \times p_3, p_2 \times p_3, p_1 \times p_2 \times p_3\}$. The minimum degree $\delta(G)$ is three and the graph has exactly four minimum degree vertices. Let the vertices be p_1, p_2, p_3 and $p_1 \times p_2 \times p_3$. We can easily observe that $N(p_1) = \{p_1 \times p_2, p_1 \times p_3, p_1 \times p_2 \times p_3\}, N(p_2) = \{p_1 \times p_2, p_2 \times p_3, p_1 \times p_2 \times p_3\}$,
$N(p_3) = \{p_1 \times p_3, p_2 \times p_3, p_1 \times p_2 \times p_3\}$ and $N(p_1 \times p_2 \times p_3) = \{p_1, p_2, p_3\}$. The removal of $N(p_1)$ or $N(p_2)$ or $N(p_3)$ or $N(p_1 \times p_2 \times p_3)$ from V_n makes the graph disconnected and hence the sets
$N(p_1), N(p_2), N(p_3), N(p_1 \times p_2 \times p_3)$ are considered as the minimum vertex cut S of G. The graph $G - N(p_1)$ contains two components namely C_1 and C_2. Clearly C_1 is an isolated vertex $p_1$and C_2 is a connected graph with $|V(C_2)| > 1$ and $|E(C_2)| \geq 1$. Suppose C_2 is not connected then it contradicts the adjacency of the arithmetic graph $G = V_n$. Thus in this case G is hyper connected and super connected. Hence
$\kappa_s(G) = |N(P_1)| \, or \, |N(P_2)| \, or \, |N(P_3)| \, or \, |N(P_1 \times P_2 \times P_3)| = 3 = \kappa_h(G)$.

Case (iii) when $r > 3$
In this case the arithmetic graph is $G = V_n$, where $n = p_1 \times p_2 \times \ldots \times p_r$. The vertex set of G is $\{p_1, p_2 \ldots, p_r, p_1 \times p_2, p_1 \times p_3, ..., p_1 \times p_r, p_2 \times p_3, .., p_1 \times p_2 \times p_3, \ldots, p_1 \times p_2 \times \ldots \times p_r\}$. To prove G is super connected. By theorem 1.3, it is easy to verify that the minimum degree of G is r and $d(p_1 \times p_2 \times \ldots \times p_r) = r$. Therefore the vertex $p_1 \times p_2 \times \ldots \times p_r$ is adjacent to exactly r vertices. By the definition of an arithmetic graph $p_1, p_2, p_3 \ldots, p_r$ are adjacent vertices of $p_1 \times p_2 \times \ldots \times p_r$. Hence the set $S = \{p_1, p_2, p_3, \ldots, p_r\}$ form a minimum vertex cut of G. Suppose $G - S$ has no isolated vertices. Then $G - S$ is a disconnected graph containing connected components. Therefore the vertex $p_1 \times p_2 \times \ldots \times p_r$ must be connected to some other vertices and hence $d(p_1 \times p_2 \times \ldots \times p_r) > r$ which is a contradiction to $d(p_1 \times p_2 \times \ldots \times p_r) = r$. This implies that G is super connected. Since $|s| = r$ we have $\kappa_s(G) = r$. Now to prove $G - S$ is hyper connected. Since G has only one minimum degree vertex $p_1 \times p_2 \times \ldots \times p_r$ and $\kappa(G) \leq \delta(G)$ the minimum vertex cut S contains exactly r vertices namely $p_1, p_2, p_3 \ldots, p_r$. Therefore $G - S$ contains an isolated vertex $p_1 \times p_2 \times \ldots \times p_r$ and a connected component. Hence G is hyper connected and the hyper connected number $\kappa_h(G) = r$. □

Theorem 2.2. *For an arithmetic graph* $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_r^{a_r}$ *where* $a_i \leq 2$ *and* $r > 3$ *is super and hyper connected.*

Proof. Consider the arithmetic graph $G = V_n$, let us classify that the vertices of G be prime vertices, prime power vertices, product of prime vertices and product of prime power vertices. Now we look into the adjacency of the vertices. Consider the vertex $v_1 = \Pi \lim_{i=1}^{r} p_i^{a_i}$, where $a_i's$ are the maximum exponent of p_i, from the vertices of the given graph G. Clearly $d(v_1) = r$. Choose $v_2 = \Pi \lim_{i=1}^{r} p_i^{b_i}$ where $b_i < a_i$ for at least one i, such that the degree of v_2 must be greater than v_1 by at least one. Since otherwise it contradicts the definition of an arithmetic graph. Therefore $d(v_1) < d(v_2)$, continuing the process we observe that $d(v_1) < d(v_2) \leq \cdots \leq d(p_r) \cdots \leq \cdots \leq d(p_1)$ (Here the equality holds for some of the vertices). We can easily say that the vertex v_1 is the only vertex with minimum degree. Hence $N(v_1)$ is the minimum vertex cut, let it be S. Clearly $G - S$ has an isolated vertex v_1. Therefore the given graph is super connected and hyper connected. Hence $\kappa_s(G) = \kappa_h(G) = r$. □

Remark 2.3. *If* $G = V_n$ *be an arithmetic graph where* $n = p_1^{a_1} \times p_2^{a_2}$,*with the order of* a_i *is not considered then*

$$n(S) = \begin{cases} 1 \text{ if } a_1 = 1, a_2 \geq 2 (or) a_1, a_2 > 2 \\ 2 \text{ if } a_1 = 2 \text{ and } a_2 > 2 \\ 3 \text{ if } a_1, a_2 = 2. \end{cases}$$

where $n(S)$ *is the number of minimum vertex cuts of G, and the order is not considered.*

Proof. The proof follows from the definition of an arithmetic graph. □

Theorem 2.4. *For an arithmetic graph* $G = V_n$, *the following holds (i)If* $n = p_1^{a_1} \times p_2^{a_2}, a_1 > 2$ *and* $a_2 = 2$,*then the number of vertices having minimum degree is* $2(a_1 - 1)$ *and the graph has two minimum vertex cuts. (ii)If* $n = p_1^{a_1} \times p_2^{a_2}; a_1, a_2 > 2$, *then the number of vertices having minimum degree is* $(a_1 - 1)(a_2 - 1)$ *and the graph has only one minimum vertex cut.*

Proof. (i) Consider the arithmetic graph $G = V_n$, where $n = p_1^{a_1} \times p_2^{a_2}; a_1 > 2$ and $a_2 = 2$. The vertex set $V(G) = \{p_1, p_1^2, p_1^3, \dots, p_1^{a_1}, p_2, p_2^2, p_1 \times p_2, p_1^2 \times p_2, \dots, p_1^{a_1} \times p_2, p_1 \times p_2^2, p_1^2 \times p_2^2, \dots, p_1^{a_1} \times p_2^2\}$. Here the vertices $\{p_1^{a_i};\ 2 \le a_i \le a_1\}$ are adjacent to exactly two vertices namely $p_1 \times p_2$ and $p_1 \times p_2^2$. Hence $d(p_1^2) = d(p_1^3) = \cdots = d(p_1^{a_1}) = 2$. Thus $(a_1 - 1)$ vertices of G have degree 2. Also the vertices $\{p_1^2 \times p_2^2, p_1^3 \times p_2^2, \dots, p_1^{a_1} \times p_2^2\}$ are adjacent to the vertices p_1 and p_2. That is $N(p_1^{a_i} \times p_2^2) = \{p_1, p_2\}$ for $2 \le a_i \le a_1$. Hence $d(p_1^2 \times p_2^2) = d(p_1^3 \times p_2^2) = \cdots = d(p_1^{a_1} \times p_2^2) = 2$. Again $(a_1 - 1)$ vertices are of degree 2. Thus the total number of vertices of degree 2 are $(a_1 - 1) + (a_1 - 1) = 2(a_1 - 1)$. Also, we can easily say that the given arithmetic graph G has two minimum vertex cuts $\{p_1, p_2\}$ and $\{p_1 \times p_2,\ p_1 \times p_2^2\}$.
(ii) If $n = p_1^{a_1} \times p_2^{a_2}$; $a_1, a_2 > 2$. The vertex set of G is $V(G) = \{p_1, p_1^2, p_1^3, \dots, p_1^{a_1}, p_2, p_2^2, p_2^3, \dots, p_2^{a_2}, p_1 \times p_2, p_1 \times p_2^2, \dots, p_1 \times p_2^{a_2}, p_1^2 \times p_2, p_1^2 \times p_2^2, \dots, p_1^2 \times p_2^{a_2}, p_1^2 \times p_2, \dots, p_1^{a_1} \times p_2, p_1^{a_1} \times p_2^2, \dots, p_1^{a_1} \times p_2^{a_2}\}$ By 1.3, $\delta(G) = 2$. Let $A = \{p_1^2, p_1^3, \dots, p_1^{a_1}\}$; $|A| = (a_1 - 1)$ and $B = \{p_2^2, p_2^3, \dots, p_2^{a_2}\}$; $|B| = (a_2 - 1)$. Clearly the cartesian product of A and B is the set $A \times B$ containing $(a_1 - 1) \times (a_2 - 1)$ vertices. By the definition of an arithmetic graph, these $(a_1 - 1) \times (a_2 - 1)$ vertices have the common neighbourhoods p_1 and p_2. Thus the vertex set $\{p_1, p_2\}$ is the only minimum vertex cut of G. □

Theorem 2.5. *In an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2}$, at least one $a_i \ge 3$ then $\kappa_h(G) = \infty$,*
$\kappa_s(G) = \Big\{ 1,$ *if* $a_1 \ge 3$ *and* $a_2 = 12,$ *if* $a_1 \ge 3$ *and* $a_2 \ge 1.$

Proof. **Case (i)**
Without loss of generality let us assume that $a_1 \ge 3$ and $a_2 = 1$. Let the vertices of G be $V(G) = \{p_1, p_2, p_1^2, p_1^3, p_1 \times p_2, p_1^2 \times p_2, p_1^3 \times p_2\}$. Clearly $p_1 \times p_2$ is the only neighbour for the vertices $\{p_1^2, p_1^3, \dots, p_1^{a_1}\}$. Hence $N(p_1^2) = N(p_1^3) = \cdots = N(p_1^{a_1}) = \{p_1 \times p_2\}$. By the result 2.3, the vertex set $S = \{p_1 \times p_2\}$ is the minimum vertex cut of the given graph G. Here $G - S$ is a disconnected graph with at least a_1 components namely the isolated vertices $\{p_1^2, p_1^3, \dots, p_1^{a_1}\}$ and a connected graph. Thus G is super connected but not hyper connected. Since S contains only one element, the super connected number $\kappa_s(G)$=1. Since the graph is not hyper connected the hyperconnected number $\kappa_h(G) = \infty$.
Case (ii)
Without loss of generality let us assume that $a_1 \ge 3$ and $a_2 = 2$. Let the vertices of G be $V(G) = \{p_1, p_2, p_1^2, p_2^2, p_1^3, p_1 \times p_2, p_1 \times p_2^2, p_1^2 \times p_2, p_1^3 \times p_2, p_1^2 \times p_2^2, p_1^3 \times p_2^2,\}$. Also in 1.3, $\delta(G) = 2$ and by result 2.3, there exists only two minimum vertex cuts of G. Since the vertices p_1^2, p_1^3, p_2^2 are adjacent only to the vertices $p_1 \times p_2$ and $p_1 \times p_2^2$, $N(p_1^2) = N(p_1^3) = N(p_2^2) = \{p_1 \times p_2, p_1 \times p_2^2\}$. Also the vertices $p_1^2 \times p_2^2, p_1^3 \times p_2^2$ are adjacent only to p_1 *and* p_2. We have $N(p_1^2 \times p_2^2, p_1^3 \times p_2^2) = \{p_1, p_2\}$. Therefore $S_1 = \{p_1 \times p_2, p_1 \times p_2^2\}$ and $S_2 = \{p_1, p_2\}$ are the two minimum vertex cuts of G and the removal of S_1 or S_2 from G makes the graph disconnected with at least three components having two isolated vertices. Hence the graph is super connected and the cardinality of the minimum vertex cut is the super connected number $\kappa_s(G)$=2. But the graph is not hyper connected so $\kappa_h(G) = \infty$
Case (iii)
Let $n = p_1^{a_1} \times p_2^{a_2}$ if both $a_i > 2$,the vertices of G be $V(G) = \{p_1, p_1^2, p_1^3, \dots, p_1^{a_1}, p_2, p_2^2, p_2^3, \dots, p_2^{a_2}, p_1 \times p_2, p_1 \times p_2^2, \dots, p_1 \times p_2^{a_2}, p_1^2 \times p_2, p_1^2 \times p_2^2, \dots, p_1^2 \times p_2^{a_2}, p_1^3 \times p_2, p_1^3 \times p_2^2, \dots, p_1^3 \times p_2^{a_2}, \dots, p_1^{a_1} \times p_2, \dots, p_1^{a_1} \times p_2^{a_2}\}$. By result 2.3, there exists only one minimum vertex cut,and by theorem 1.4 let it be $S = \{p_1, p_2\}$ but there are more than one vertex which are adjacent only to p_1 and p_2. Hence $G - S$ is a disconnected graph with at least two isolated vertices and a connected component.Therefore the graph is super connected but not hyper connected. Hence $\kappa_s(G) = |S| = 2$ and $\kappa_h(G) = \infty$ □

Theorem 2.6. *In an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$ at least one $a_i \ge 3$ then G is super connected but not hyper connected.*

Proof. Consider the arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$ at least one $a_i \ge 3$. In[1],$|V(G)| = [(a_1 + 1)(a_2 + 1) \dots (a_r + 1)] - 1$ and $\delta(G) = r$. Let $v_1 = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$ be the vertex of G with higher exponent a_i and arrange $a_i' s$ such that $a_1 \ge a_2 \ge a_3 \cdots \ge a_r$. Here $d(v_1) = r$. Let us consider the vertex $v_2 = p_1^{b_1} \times p_2^{b_2} \times \cdots \times p_r^{b_r}, b_1 \ge b_2 \ge b_3 \cdots \ge b_r$ where $b_1 = a_1 - 1$ and $b_i = a_i$ for every $i = 1, 2, \dots, r$ also $d(v_2) = r$. Thus there exists atleast two vertices of minimum degree with same neighbourhood and the set, $S = \{p_1, p_2, \dots, p_r\}$ is the minimum vertex cut of G. Hence the cardinality of S is the super connected number. Thus $\kappa_s(G) = r$. Since $G - S$ has more than two isolated vertices the graph is not hyper connected hence the hyper connected number is $\kappa_h(G) = \infty$ □

Remark 2.7. *In an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$ $a_i \le 2$ the super connected and hyper connected number are equal to its connecivity number $\kappa(G)$.*
$(i.e)\kappa_s(G) = \kappa_h(G) = \kappa(G)$.

Remark 2.8. *In an arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$ at least one $a_i > 2$ then $\kappa_s(G) = \kappa(G) \ne \kappa_h(G)$.*

Theorem 2.9. *Every arithmetic graph $G = V_n, n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}$ is super and hyper edge connected.*

Proof. The proof is obvious from the definition and above results □

3. Super Connectivity of an arithmetic graph $\kappa_{sc}(G)$

The definition of a super connectivity number of a graph is studied from [3].The authors Jun-Ming Xu, Min Lu, Meijie Ma, Angelika Hellwig used the definition for line graphs. We studied the concept in arithmetic graphs.
The following steps are used to find the super connectivity number $\kappa_{sc}(G)$ of an arithmetic graph $G = V_n$. Let $G = V_n$ be an arithmetic graph where $n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}, a_i \ge 1$ and P_i^s are distinct primes . Let the vertex set of G be $V(G) =$

$\{v_1, v_2, v_3, \dots v_n\}$.
step1:
Choose two vertices u and v in $V(G)$ such that $d(u) + d(v) = min\{d(v_i) + d(v_j) | v_i v_j \in E(G), i \neq j, i, j = 1, 2...r\}$
step2:
Take $X = N(u) - \{v\}$ and $Y = N(v) - \{u\}$.
step3:
Find T and S such that T is the set of vertices which are adjacent only to X and Y, and $S = \{T \cup X \cup Y\}$.
step4:
Observe $G - S$, If every component of $G - S$ contains no isolated vertices then S is a super vertex cut and $|S|$ is a super connectivity number .If not, S is not a super connectivity set of G and the super connectivity number is ∞.

Theorem 3.1. *In an arithmetic graph $G = V_n$, $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_r^{a_r}$ where $a_i = 1$ for every i*
then $\kappa_{sc}(G) = \begin{cases} \infty \, for \;\; r \leq 3 \\ a_i \prod \lim_{j=1, i \neq j}^{r} (a_j + 1) + r - 2 \;\; for \;\; r > 3 \end{cases}$

Proof. **Case (i)** When $r \leq 3$
Sub case (i) If $r = 2$ then the arithmetic graph is a tree with three vertices so the internal vertex is a cut vertex, say v and the graph $G - v$ has two isolated vertices. Hence G has no super vertex cut. Therefore by definition $\kappa_{sc}(G) = \infty$.
Sub case (ii) If $r = 3$ then, it is clear that G contains exactly three minimum vertex cuts namely $S_1 = \{P_1 \times P_2 \times P_3, P_1 \times P_2, P_1 \times P_3\}, S_2 = \{P_1, P_2, P_3\}$ and $S_3 = \{P_1 \times P_2 \times P_3, P_2 \times P_3, P_1 \times P_3\}$. Also $G - S_1$ has two components such as an isolated vertex and a connected component which is an arithmetic graph $P_1 \times P_2$. Therefore S_1 is not a super vertex cut of G . If we include the isolated vertex to S_1,the graph $G - \{S \cup P_1\}$ is a connected graph $P_1 \times P_2$,and hence by sub case(i)there does not exist any super vertex cut. Similarly for other vertex cuts S_2 and S_3.
Case (ii) When $r > 3$
Using the above procedure we can choose the two vertices as $v_1 = P_i$ for any $i = 1, 2, 3, \dots r$ and $v_2 = P_1 \times P_2 \times P_3 \times \cdots \times P_r$. The graph $G - S$ has exactly two components namely the edge $v_1 v_2$ and a connected component containing the vertices which are the combinations of product of 2 primes,3 primes,$\dots$,$(r-2)$ primes in the vertex set $\{P_1, P_2, P_3 .....P_{i-1}, P_{i+1}, \dots, P_r\}$ and the adjacency according to the definition of an arithmetic graph. Since each component has no isolated vertices, by the definition, S is a super vertex cut.
Therefore $\kappa_{sc}(G) = |S|$
$= |X| + |N(v_1) - v_2| + |N(v_2) - v_1|$
$= 1 + a_i \prod \lim_{j=1, i \neq j}^{r} (a_j + 1) - 2 + r - 1$
$= a_i \prod \lim_{j=1, i \neq j}^{r} (a_j + 1) + r - 2$ ☐

Theorem 3.2. *For an arithmetic graph $G = V_n$,$n = P_1^{a_1} \times P_2^{a_2}, a_1 > 1, a_2 = 1$ then $\kappa_{sc}(G) = \infty$.*

Proof. Consider the arithmetic graph $G = V_n$ where $a_1 = r > 1; a_2 = 1$. For every $v \in V(G)$

$$d(v) = \begin{cases} r \; if \; v = P_1 \; or \; P_2 \\ r+1 \; if \; v = P_1 \times P_2 \\ 1 \; if \; v = P_1^k, k = 2 \dots r \\ 2 \; if \; v = P_1^k \times P_2, k = 2 \dots r \end{cases}$$

By the definition of arithmetic graph and the above procedure we can easily observe that the two adjacent vertices whose degree sum is minimum are $P_1 \times P_2$ and P_1^k or P_i;i=1,2 and $P_1^k \times P_2; k = 2, 3, ..r$.By the procedure it is clear that there does not exist any super vertex cut for G. Hence $\kappa_{sc}(G) = \infty$. ☐

Theorem 3.3. *For an arithmetic graph $G = V_n, n = P_1^{a_1} \times P_2^{a_2}, a_1, a_2 > 1$ then $\kappa_{sc}(G) = a_1 a_2$.*

Proof. By theorem 3.3 and the procedure we can observe that adjacent vertices whose degree sum is minimum are $P_1^{a_1}, a_1 > 1$ and $P_1 \times P_2^{a_2}, a_2 > 1$ or $P_2^{a_2}$ and $P_1^{a_1} \times P_2, a_1, a_2 > 1$
Therefore, $\kappa_{sc}(G) = d(P_1^{a_1}) + d(P_1 \times P_2^{a_2}) - 2 + (a_1 - 1)(a_2 - 1)$
$= a_1 + a_2 + 1 - 2 + (a_1 - 1)(a_2 - 1)$
$= a_1 + a_2 - 1 + (a_1 - 1)(a_2 - 1)$
$= a_1 + (a_2 - 1)(1 + a_1 - 1)$
$= a_1 + a_1(a_2 - 1)$
$= a_1 a_2$. ☐

4. Hyper connectivity of an arithmetic graph

The definition of a hyper connectivity number of a graph is a set $S \subset V(G)$ is called a hyper vertex cut if $G - S$ is not connected and
(i)Each component of $G - S$ contains no isolated vertices
(ii)Exactly one component of $G - S$ is K_2. The hyper connectivity $\kappa_{hc}(G)$ is the minimum cardinality over all hyper vertex cuts in G. If there doesnot exist such S then $\kappa_{hc}(G) = \infty$.

Remark 4.1. *We used a procedure for finding the super conneced number of an arithmetic graph. Similarly the following steps are used to find the hyper connectivity number $\kappa_{hc}(G)$ of an arithmetic graph $G = V_n$.*
Let $G = V_n$ be an arithmetic graph where $n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_r^{a_r}, a_i \geq 1$ and P_i^s are distinct primes. Let the vertex set of G be $V(G) = \{v_1, v_2, v_3, \dots v_n\}$.
Step 1,2,3 follows from procedure 2.1
Step.4:Observe $G - S$,if everycomponent of $G - S$ are not isolated vertices and exactly one component is K_2 then S is a hyper vertex cut and $|S|$ is a hyper connectivity number. If not,S is not a hyper vertex cut of G and hence the hyper connectivity number is ∞.

Theorem 4.2. *In an arithmetic graph $G = V_n, n = P_1^{a_1} \times P_2^{a_2}$, if at least one $a_i = 1, i = 1, 2$;then $\kappa_{hc}(G) = \infty$.*

Proof. **Case (i)** If both $a_i = 1, i = 1, 2$;then by theorem 2.1, the graph does not satisfy hyper connecivity property. Hence $\kappa_{hc}(G) = \infty$.
Case (ii) If any one of the $a_i = 1$ and the other greater than

one then by theorem 1.6,G is a bipartite graph in which one of the partition contains exactly two vertices of degree greater than one andall the other vertices are of degree one.By applying procedure2,the set $S=\{P_2, P_1\times P_2\}$ makes the graph disconnected with exactly one component is K_2 and the other components are isolated vertices.Thus,by the definition 3.1, $\kappa_{hc}(G)=\infty$. □

Theorem 4.3. *For an arithmetic graph* $G=V_n, n=P_1^{a_1}\times P_2^{a_2}$,*$P_1$ and P_2 are distinct primes and* $a_1=a_2=2$; *then* $\kappa_{hc}(G)=\infty$.

Proof. Clearly the given arithmetic graph G contains 8 vertices.Let the vertex set $V(G)$ be $\{P_1, P_2, P_1^2, P_2^2, P_1\times P_2, P_1^2\times P_2, P_1\times P_2^2, P_1^2\times P_2^2\}$ By applying the procedure,we get the set $S=\{P_1, P_2, P_1\times P_2\}$.Clearly $G-S$ has exactly three components in which two components are k_2 and the third component is an isolated vertex.Hence we get $\kappa_{hc}(G)=\infty$. □

Theorem 4.4. *For an arithmetic graph* $G=V_n, n=P_1^{a_1}\times P_2^{a_2}$ *where* $a_1>2, a_2\geq 2$;*then* $\kappa_{hc}(G)=a_1a_2$.

Proof. It is clear that by theorem 1.6 the arithmetic graph $G=V_n, n=P_1^{a_1}\times P_2^{a_2}; a_1>2, a_2\geq 2$ is a bipartite graph.Let X_1, X_2 be the partitions such that $X_1=\{P_1, P_1^2, \ldots P_1^{a_1}, P_2, P_2^2, \ldots P_2^{a_2}\}$ and $X_2=\{P_1\times P_2, P_1\times P_2^2, \ldots, P_1\times P_2^{a_2}, P_1^2\times P_2, P_1^2\times P_2^2, \ldots, P_1^2\times P_2^{a_2}, P_1^3\times P_2, P_1^3\times P_2^2\ldots P_1^3\times P_2^{a_2}, \ldots P_1^{a_1}\times P_2, P_1^{a_1}\times P_2^2, \ldots, P_1^{a_1}\times P_2^{a_2}\}$.By theorem1.5,the vertex cut of G is $S=\{P_1\times P_2, P_1\times P_2^2, \ldots, P_1\times P_2^{a_2-1}, P_1, P_1^2, \ldots, P_1^{a_1-1}, P_1^{a_1}\times P_2^{a_2}, a_1>1, a_2>1\}$. Since $G-S$ satisfies the requirement of hyper connectivity property,we have S is the hyper vertex cut of G and $\kappa_{hc}(G)=|S|=a_2+a_1+1+(a_1+1)(a_2+1)-2=a_1a_2$. □

Theorem 4.5. *For an arithmetic graph* $G=V_n, n=p_1^{a_1}\times p_2^{a_2}\times\ldots\times p_r^{a_r}$ *where $a_i=1$ for every i then*

$$\kappa_{hc}(G)=\begin{cases}\infty \;\; for \;\; r\leq 3\\ a_i\prod\lim_{j=1,i\neq j}^{r}(a_j+1)+r-2\, for \;\; r>3\end{cases}$$

Proof. The theorem follows from theorem 3.1. □

Theorem 4.6. *For an arithmetic graph* $G=V_n, n=p_1^{a_1}\times p_2^{a_2}\times\ldots\times p_r^{a_r}$ *where $a_i>1$ for at least one* $i, i=1,2,\ldots,r$ *then* $\kappa_{hc}(G)=\kappa_{sc}(G)=$
$\left[|B|\prod\lim_{i=1,i\notin B}^{r}(a_i+1)-1\right]+[|B-B'|+\sum\lim_{i\in B'}a_i]\prod\lim_{i=1,i\notin B}^{r}(a_i+1)-2+n+m$, *where $B\subseteq\{1,2,r\}$ and n denote the number of vertices of degree r and* $p_1^2\times p_2^2\times\ldots\times p_r^2\leq m\leq p_1^{\alpha_1}\times p_2^{\alpha_2}\times\ldots\times p_r^{\alpha_r}, 2\leq\alpha_i\leq a_i; i=1,2,..r$.

Proof. Using the procedure choose two adjacent vertices whose degree sum is minimum are of the form,$p_i^{a_i}, a_i>1$ and $p_1^{a_1}\times p_2^{a_2}\times\ldots\times p_r^{a_r}$ where $(a_i=1)\,1\leq a_j\leq\alpha_j, (j\neq i); j=1,2,i-1,i+1,\ldots r$. Applying step 3 of the procedure, we get the vertex cut S consist of $N\left(p_i^{a_i}\right), a_i>1$ and $N\left(p_1^{a_1}\times p_2^{a_2}\times\ldots\times p_r^{a_r}\right), a_i=1$.Clearly the graph $G-S$ contains $n+m$ isolated vertices where $n=|S_1|$ the number of vertices having degree r in G and $m=|S_2|$ the number of vertices which are of the form $p_1^2\times p_2^2\times\ldots\times p_r^2\leq m\leq p_1^{\alpha_1}\times p_2^{\alpha_2}\times\ldots\times p_r^{\alpha_r}, 2\leq\alpha_i\leq a_i; i=1,2,..r$.Clearly S is not a hyper vertex cut. But $S_3=S\cup S_1\cup S_2$ is a hyper vertex cut of G having exactly one k_2 and a connected component having more than two vertices in $G-S_3$.Therefore $\kappa_{hc}(G)=|S_3|$. □

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